Periodic orbits, breaktime and localization

1998 J. Phys. A: Math. Gen. 31 277

(http://iopscience.iop.org/0305-4470/31/1/025)

J. Phys. A: Math. Gen. **31** (1998) 277–287. Printed in the UK PII: S0305-4470(98)85907-6

Periodic orbits, breaktime and localization

Doron Cohen

Department of Physics of Complex Systems, The Weizmann Institute of Science, Rehovot 76100, Israel

Received 11 July 1997

Abstract. The main aim of this paper is to realize that it is feasible to construct a 'periodic orbit theory' of localization by extending the idea of classical action correlations. This possibility had been questioned by many researchers in the field of 'quantum chaos'. Starting from the semiclassical trace formula, we formulate a quantal-classical duality relation that connects the spectral properties of the quantal spectrum to the statistical properties of lengths of periodic orbits. By identifying the classical correlation scale it is possible to extend the semiclassical theory of spectral statistics, in case of a complex systems, beyond the limitations that are implied by the diagonal approximation. We discuss the quantal dynamics of a particle in a disordered system. The various regimes are defined in terms of time-disorder 'phase diagram'. As expected, the breaktime may be 'disorder limited' rather than 'volume limited', leading to localization if it is shorter than the ergodic time. Qualitative agreement with scaling theory of localization in one to three dimensions is demonstrated.

1. Introduction

Extending the semiclassical approach to spectral statistics beyond the diagonal approximation is presently one of the most vigorously pursued directions of research in 'quantum chaos'. It is desirable to reach a semiclassical understanding of the long-time behaviour also for disordered systems. They play a central role in condensed-matter as well as in mesoscopic physics. The introduction of semiclassical methods in the latter case is quite natural. It can be expected that a semiclassical insight into localization will, in turn, shed new light on semiclassical methods in general. This paper is intended as a contribution towards this goal. It rests mainly on two previous observations: the connection between spectral correlations in the long-time regime and classical action-correlations [1], and the heuristic treatment of localization by Allen [2]. It turns out that the latter appears as a natural consequence of the former, once a disorder system is considered. An improved qualitative picture of spectral statistics follows, expressed in the form of a 'time-disorder' diagram. Furthermore, the present formulation paves the way towards a quantitative account in terms of the spectral form factor.

There are few timescales that are associated with the semiclassical approximation for the time evolution of any observable. Such an approximation involves a double-summation over classical orbits. The purpose of the following paragraphs is to make a clear distinction between these various timescales. In particular, we wish to clarify the 'breaktime' concept that plays a central role in our formulation. Initially the classical behaviour is followed. One relevant timescale for the departure from the classical behaviour is $t_{\rm scl}$. By definition, when $t_{\rm scl} < t$ deviations that are associated with the breakdown of the *stationary phase approximation* may show up. It has been argued [3] that $t_{\rm scl} \sim \bar{h}^{-1/3}$. Further deviations

0305-4470/98/010277+11$19.50

from the leading-order semiclassical expansion due to diffraction effects are discussed in [4]. One should be careful not to confuse these deviations, which are associated with the accuracy of the stationary phase approximation with the following discussion of the breaktime concept. It is assumed in the sequel that the leading-order semiclassical formalism constitutes a qualitatively good approximation also for $t_{\rm scl} < t$ despite these deviations.

Interference effects lead to further deviations from the classical behaviour. Well isolated classical paths, for which the stationary phase approximation is completely accurate, may still give rise to either constructive or destructive interference effects. Hereafter we shall focus on the semiclassical computation of the spectral form factor [5], where the double summation is over classical *periodic orbits* (POs). We shall disregard extremely short times, for which only few POs contribute, since for any generic chaotic system the POs proliferate exponentially with time. The simplest assumption would be that the interference contribution (off-diagonal terms) is self-averaged to zero. However, such an assumption would imply that the classical behaviour is followed for arbitrarily long times. This is obviously not true. After a sufficiently long time the discrete nature of the energy spectrum becomes apparent, and the *recurrent* quasiperiodic nature of the dynamics is revealed. The breaktime t^* is the timescale which is associated with the latter crossover. Neglecting the interference contribution for $t < t^*$ is known as the diagonal approximation [5].

From a semiclassical point of view the breaktime t^* is related to the breakdown of the *diagonal approximation*. It has been conjectured that the breakdown of the diagonal approximation is a manifestation of classical action correlations [1]. Otherwise, if the actions were uncorrelated (Poisson statistics), then the off-diagonal (interference) contribution would be self-averaged to zero. Typically the breaktime t^* is identified with the Heisenberg time $t_H = 2\pi\hbar/\Delta E$, where ΔE is the average level spacing. The Heisenberg time is semiclassically much longer than the 'log' time $t_E \sim \ln(1/\hbar)$ over which classical orbits proliferate on the uncertainty scale. The latter timescale has no physical significance as far as the form factor is concerned. (See also the discussion after equation (12).)

The breaktime which is determined by the Heisenberg uncertainty relation is volume dependent. However, for a disordered system the breaktime may be much shorter and volume independent due to the localization effect. The theory for this 'disordered-limited' rather than 'volume-limited' breaktime constitutes the main theme of this paper. Our approach to deal with disorder within the framework of the semiclassical approach constitutes a natural extension of previous attempts to integrate 'mesoscopic physics' with the so-called field of 'quantum chaos', (see [6] for a review). Note that a naive semiclassical arguments can be used in order to estimate the breaktime and localization length for one-dimensional systems [7]. (See the next section for further details.) This argument, as it stands, cannot be extended to higher dimensions, which implies that a fundamentally different approach is needed. The same objection applies to a recent attempt to propose a periodic orbit theory for one-dimensional localization [8]. In the latter reference the semiclassical argument for localization is based on proving exponentially small sensitivity for change in boundary conditions. This is due to the fact that only an exponentially small number of POs with $t < t_H$ hit the edges. The statement holds for $d = 1$, where $t_H \ll t_{\rm erg}$, but it fails in higher dimensions. Hence, the necessary condition for having localization should be much weaker.

This paper is organized as follows. The expected results for the disordered-limited breaktime, based on scaling theory of localization, are presented in section 2. Our main goal is to rederive these results from semiclassical consideration. In section 3, starting from the semiclassical trace formula (SCTF), we formulate a duality relation that connects the spectral properties of the quantal spectrum to the classical two-point statistics of the POs.

In section 4 we identify the classical correlation scale. Then it is possible to extend the semiclassical theory of spectral statistics, in the case of a complex systems, beyond the limitations that are implied by the diagonal approximation. In section 5 we demonstrate that a disorder-limited breaktime is indeed a natural consequence of our formulation. The various time regimes for a particle in a disordered system are illustrated using a time-disorder 'phase diagram'. Localization shows-up if there is a disorder-limited breaktime which is shorter than the ergodic time. Semiclassical interpretation for the existence of critical and ohmic regimes for three-dimensional localization is also introduced. Finally, in section 6, we introduce a semiclassical approximation scheme for the form factor, that goes beyond the diagonal approximation. The limitations of this new scheme are pointed out.

Effects that are associated with the actual presence of a magnetic field are not considered in this paper, since the SCTF should then be modified. Still, for simplicity of presentation we cite for the form factor the GUE rather than the GOE result, and we disregard the effect of time reversal symmetry. A proper treatment of these details is quite obvious, and will appear in a future publication [9]. It is avoided here in order not to obscure the main point.

2. Breaktime for disordered systems

We consider a particle in a disordered potential. The classical dynamics is assumed to be diffusive. For concreteness we refer to a *disordered billiard*. The concept is defined below. It should be emphasized that we assume genuine disorder. Pseudorandom disorder, as well as spatial symmetries are out of the scope of our considerations.

A disordered billiard is a quasi-d-dimensional structure that consists of connected chaotic cavities. Here we summarize the parameters that are associated with its definition. The billiard is embedded in d_0-dimensional space ($2 \leqslant d_0$). It constitutes a d-dimensional structure of cells (obviously $d \leqslant d_0$). Each cell, by itself, constitutes a chaotic cavity whose volume is roughly $\ell_0^{d_0}$. However, the cells are connected by small holes whose area is $a_0^{d_0-1}$ with $a_0 \ll \ell_0$. The volume of the whole structure is $\Omega = \mathcal{L}^d \ell_0^{d_0-d}$. Assuming a classical particle whose velocity is v, the average escape time out of a cell is $t_0 \approx (\ell_0/v) \cdot (\ell_0/a_0)^{d_0-1}$. The classical diffusion coefficient is $\mathcal{D}_0 = \ell_0^2/t_0$. The classical diffusion law is $\langle (x - x_0)^2 \rangle = \mathcal{D}_0 t$ where x_0 is the location of an initial distribution.

The mass of the particle is m. Its de Broglie wavelength $\lambda_{\mathrm{B}} = \hbar/mv$ is assumed to be much shorter than ℓ_0 as to allow (later) semiclassical considerations. Actually, in order to have non-trivial dynamical behaviour λ_{B} should be smaller than or at most equal to a_0 (note the following definition of the dimensionless conductance). The mean energy level spacing is $\Delta E = 2\pi\alpha^{-1} \cdot (\hbar^{d_0}/\mathcal{L}^d)$ where $\alpha \sim \ell_0^{d_0-d} m^{d_0-1} v^{d_0-2}$. The Heisenberg time is $t_H = 2\pi\hbar/\Delta E = \alpha \mathcal{L}^d/\hbar^{d_0-1}$ The dimensionless conductance g_0 on scale of *one* cell is defined as the ratio of the Thouless energy $2\pi\hbar/t_0$ to the level spacing $2\pi\hbar/t_H$. (for t_H one should substitute here $\mathcal{L} = \ell_0$). Hence $g_0 = (a_0/\lambda_B)^{d_0-1}$ is simply related to the hole size. Out of the eight independent parameters $(d_0, d, \ell_0, a_0, \mathcal{L}, m, v, \hbar)$ there are actually only three dimensionless parameters which are relevant. Setting t_0 and ℓ_0 to unity, these are d, g_0 and $\mathcal{L}$. All the results should be expressed using these parameters.

For a billiard system in d-dimension, whose volume is Ω, the Heisenberg-time is given by the expression $t_H = \alpha\Omega/\hbar^{d-1}$. For the *disordered billiard* Heisenberg time can be expressed in terms of the unit-cell dimensionless conductance

$$t_H^* = \frac{2\pi\hbar}{\Delta E} = \alpha \frac{\Omega}{\hbar^{d_0-1}} = \left(\frac{\mathcal{L}}{\ell_0}\right)^d g_0 t_0. \tag{1}$$

The actual 'disordered-limited' breaktime may be much shorter due to the localization

effect. Naive reasoning concerning wavepacket dynamics leads to the volume-independent estimate [7]

$$t^* = \frac{2\pi\hbar}{\Delta_\xi} = \alpha\frac{\xi^d}{\hbar^{d_0-1}} \qquad \text{naive.} \tag{2}$$

Here Δ_ξ is the effective level spacing within a volume ξ^d. Assuming that up to t^* the spreading is diffusive-like, it follows that

$$\xi^2 = \mathcal{D}_0 t^*. \tag{3}$$

Combining these two equations it has been argued [7] that for quasi one-dimensional structure ($d = 1$) the localization length is $\xi \sim \alpha\mathcal{D}_0/\hbar^{d_0-1}$. In terms of the dimensionless conductance, the result is $\xi = g_0\ell_0$ which corresponds to the breaktime

$$t^* = g_0^2 t_0 \qquad \text{for } d = 1. \tag{4}$$

The above argument that relates ξ and t^* to the dimensionless conductance g_0 cannot be extended in case of $1 < d$. This is due to the fact that (2) overestimates the breaktime. From scaling theory of localization [10] one obtains for $d = 2$ the result $\xi = \mathrm{e}^{g_0}\ell_0$ leading (via equation (3)) to

$$t^* = \mathrm{e}^{2g_0} t_0 \qquad \text{for } d = 2. \tag{5}$$

For $d = 3$ and $g < g_c$, where g_c is the critical value of g, scaling theory predicts $\xi = |g - g_c|^{-\nu}\ell_0$, with $\nu \approx 1/(d-2)$. Here the diffusive-like behaviour up to t^* is replaced by an anomalous scale-dependent diffusive behaviour, leading to the relation $\xi^d = \ell_0^{d-2}\mathcal{D}_0 t^*$ rather than (3), and hence

$$t^* = \frac{1}{|g_0 - g_c|^{\nu d}} t_0 \qquad \text{for } d = 3. \tag{6}$$

It is easily verified that the naive formula (2) overestimates the actual breaktime by a factor of g_0 for both $d = 2$ and $d = 3$. We turn now to develop a semiclassical theory for the breaktime.

3. Quantal-classical duality

The SCTF [11] relates the quantal density of states to the classical density of POs. The quantal spectrum $\{k_n\}$ for a simple billiard in d dimensions is defined by the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ with the appropriate boundary conditions. The corresponding quantal density is

$$\rho_{q\mathrm{m}}(k) \equiv \sum_n 2\pi\,\delta(k - k_n)|_{\mathrm{osc}}. \tag{7}$$

In order to facilitate the application of Fourier transform conventions a factor 2π has been incorporated and let $\rho(k) = \rho(-k)$ for $k < 0$. The subscript osc implies that the averaged (smoothed) density of states is subtracted. This smooth component equals the corresponding Heisenberg length and is found via Weyl law, namely

$$L_H(k) = C_d\Omega k^{d-1} \tag{8}$$

where Ω is the volume of the billiard, and $C_d = (2^{d-2}, \pi^{d/2-1}\Gamma(d/2))^{-1}$. For billiard systems, actions lengths and times are trivially related by constant factors and therefore can be used interchangeably. In the sequel some of the formulae become more intelligible if

one recalls that L actually plays the role of the time. The classical spectrum $\{L_j\}$ consists of the lengths of the POs and their repetitions. The corresponding weighted density is

$$\rho_{\rm cl}(L) \equiv \sum_j A_j \delta(L - L_j)|_{\rm osc}. \tag{9}$$

Here A_j are the instability amplitudes. We note that for a simple chaotic billiard, due to ergodicity $K_D(L) \equiv \langle \sum_j |A_j|^2 \delta(L - L_j)\rangle \sim L$. The instability amplitudes decay exponentially with L, namely $|A_j|^2 \sim L^2 \exp(-\sigma L)$, where σ is the Lyapunov exponent. Hence, the density of POs grows exponentially as $\exp(\sigma L)/L$. With the above definitions the SCTF is simply

$$\rho_{\rm qm}(k) = \mathcal{FT}\rho_{\rm cl}(L). \tag{10}$$

Where the notation $\mathcal{FT}$ is used in order to denote a Fourier transform. Both the SCTF and the statistical relation (11) that follows, reflect the idea that the quantal spectrum and the classical spectrum are two dual manifestations of the billiard boundary.

The two-point correlation function of the quantal spectrum is $R_{\rm qm}(k, \epsilon) \equiv \langle \rho_{\rm qm}(k)\rho_{\rm qm}(k+\epsilon)\rangle$, where the angle brackets denote statistical averaging. The spectral form factor $K_{\rm qm}(k, L)$ is its Fourier transform in the variable $\epsilon \rightsquigarrow L$. Due to the self-correlations of the discrete energy spectrum $R_{\rm qm}(\epsilon)$ is delta-peaked in its origin. As a consequence the asymptotic behaviour of the spectral form factor is $K_{\rm qm}(k, L) = L_H(k)$ for $L_H(k) \ll L$. For a *simple* ballistic billiard the crossover to the asymptotic behaviour occurs indeed at the Heisenberg time. The functional form of the crossover is described by random matrix theory (RMT). For concreteness we cite the approximation $K_{\rm qm}(k, L) = \min(L, L_H(k))$. (The effect of symmetries is ignored for sake of simplicity.) In order to formulate a semiclassical theory for the form factor it is useful to define the two-point correlation function of the classical spectrum $R_{\rm cl}(x, L) \equiv \langle \rho_{\rm cl}(L)\rho_{\rm cl}(L+x)\rangle$. The corresponding form factor $K_{\rm cl}(k, L)$ is obtained by Fourier transform in the variable $x \rightsquigarrow k$. It is straightforward to prove that due to the SCTF $R_{\rm qm}(k, \epsilon)$ is related to $R_{\rm cl}(x, L)$ by a double Fourier transform. Hence

$$K_{\rm qm}(k, L) = K_{\rm cl}(k, L) \tag{11}$$

which is the two-point version of the SCTF. It constitutes a concise semiclassical relation that expresses the statistical implication of quantal-classical duality. It is essential to keep the spectral form factor unrescaled. Its parametric dependence should not be suppressed. If regarded as a function of L, the quantity $K(k, L)$ is the quantal form factor, while if regarded as a function of k it is the classical form factor.

4. Beyond the diagonal approximation

The two-point statistics of the quantal density reflects the discrete nature of the quantal spectrum, and also its rigidity. It follows that the classical spectrum should be characterized by non-trivial correlations that can actually be deduced from (11). This type of argumentation has been used in [1] and will be further developed here. It is useful to write $R_{\rm cl}(x, L) = K_D(L)(\delta(x) - p(x))$ where a non-vanishing $p(x)$ implies that the classical spectrum is characterized by non-trivial correlations. Note that a proper treatment of time reversal symmetry is avoided here. Denoting the classical correlation scale by $\lambda(L)$ it follows that $K(k, L)$ should have a breaktime that is determined via $k \sim 2\pi/\lambda(L)$. For a *simple* ballistic billiard this should be equivalent to $L \sim L_H(k)$. Thus, we deduce that the classical correlation scale is

$$\lambda(L) = 2\pi \left(C_d \frac{\Omega}{L} \right)^{\frac{1}{d-1}}. \tag{12}$$

If $2\pi/\lambda \ll k$ then $K(k,L) \approx K_D(L)$, which is the diagonal approximation. More generally $K(k,L) = C(k,L)K_D(L)$, where $C(k,L) = (1-\tilde{p}(k))$ and $\tilde{}$ denotes a Fourier-transformed density. Note that it is implicit that both $p(x)$ and $\tilde{p}(k)$ depend parametrically on L. For $L_H(k) \ll L$ one should obtain the correct asymptotic behaviour $K(k,L) = L_H(k)$. Therefore $C(k,L) \to 0$ in this regime and consequently the normalization $\int_{-\infty}^{+\infty} p(x)\,\mathrm{d}x = 1$ should be satisfied. It is natural to introduce a scaling function such that $p(x) = \lambda^{-1}\hat{p}(x/\lambda)$ and consequently $C(k,L) = \hat{C}(k\lambda(L))$. For a simple ballistic billiard, neglecting modifications due to time reversal symmetry, the scaling function $\hat{C}_{\text{ballistic}}(\kappa) = \min((\kappa/2\pi)^{d-1}, 1)$ will generate the correct quantum-mechanical result. The related scaling function $\hat{p}(s)$ can be deduced via inverse Fourier transform of $(1-\hat{C}(\kappa))$.

It should be clear that the actual quantum-mechanical breaktime is related to the breakdown of the diagonal approximation. This breaktime is determined by the condition $k\lambda(L) \sim 2\pi$. If one confused $\lambda(L)$ with the classical spacing $\Delta L \sim L\exp(-\sigma L)$, then one would deduce a false breaktime at the 'log' time $t_E \sim \ln(k)$.

A heuristic interpretation of the classical two-point statistics is in order. The normalization of $p(x)$ implies rigidity of the classical spectrum on large scales. Expression (12) for the correlation scale is definitely not obvious. Still, the length scale λ possess a very simple geometrical meaning. It is simply the typical distance between neighbouring points where the PO had hit the billiard surface. It is important to notice that λ is much larger than the average spacing of the classical spectrum. The latter is exponentially small in L due to the exponential proliferation of POs. This fact suggests that the overwhelming majority of POs is uncorrelated with a given reference PO. The POs that effectively contribute to $p(x)$ must be geometrically related in some intimate way. Further discussion of these heuristic observations will be published elsewhere [9].

For a billiard that is characterized by a complicated structure, the ergodic time is much larger than the ballistic time. Orbit whose period L_j is less than the ergodic time will not explore the whole volume of the billiard but rather a partial volume Ω_j. It is quite obvious that POs that do not explore the *same* partial volume cannot be correlated in length, unless some special symmetry exists. The possibility to make a classification of POs into statistically independent classes constitutes a key observation for constructing an approximation scheme that goes beyond the diagonal approximation. Due to the classification, the spectral form factor can be written as a sum $K(k,L) = \sum_{\Omega} K(k,L,\Omega)$ of statistically independent contributions, where $K(k,L,\Omega)$ involves summation over POs with $\Omega_j \sim \Omega$. Thus, the following semiclassical expression is obtained

$$K(k,L) = \sum_j \hat{C}(k\lambda_j)|A_j|^2\delta(L-L_j) \tag{13}$$

with λ_j that corresponds to the explored volume Ω_j and with scaling function $\hat{C}(\kappa)$ that may depend on the nature of the dynamics. This formula constitutes the basis for our theory.

5. Theory of disordered billiards

We now turn to apply semiclassical considerations concerning the dynamics of a particle in a disordered system. The classical dynamics is assumed to be diffusive and we again refer to the *disordered billiard* of section 2. It should be re-emphasized that we assume genuine disorder. Pseudorandom disorder, as well as spatial symmetries may require a more sophisticated theory of PO correlations.

Hereafter we translate lengths into times by using $L = vt$. The diagonal sum over the POs satisfies $K_D(t) = tP_{\text{cl}}(t)$, where $P_{\text{cl}}(t)$ is the classical 'probability' to return [12]. For

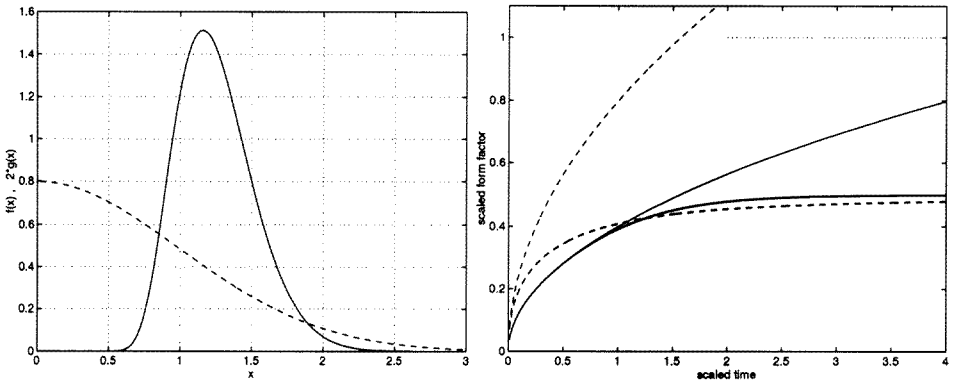

Figure 1. Left plot: The scaled probability distribution of the explored volume for a disordered chain (full curve), compared with a Gaussian distribution that characterizes the diffusion profile (broken curve). Right plot: The scaled form factor for a disordered infinite chain. Full curve—GUE result, broken curve—GOE result. The lighter curves are obtained by employing the diagonal approximation, while the heavier curves are obtained by employing the BLC approximation scheme. The dotted curve illustrates the correct asymptotic behaviour.

ballistic billiard $P_{\rm cl}(t) = 1$. This is also true for diffusive systems provided $t_{\rm erg} < t$, where

$$t_{\rm erg} = \frac{\mathcal{L}^2}{\mathcal{D}_0} = \left(\frac{\mathcal{L}}{\ell_0}\right)^2 t_0. \tag{14}$$

For $t < t_{\rm erg}$ the classical probability to return is $P_{\rm cl}(t) = (t_{\rm erg}/(2\pi t))^{d/2}$. The latter functional form reflects the diffusive nature of the dynamics.

The POs of a the disordered billiard can be classified by the volume Ω_j which they explore. By definition Ω_j is the total volume of those cells that were visited by the orbit. Let us consider POs whose length is t. Their probability distribution with respect to the explored volume will be denoted by $f_t(\Omega)$. This distribution can be deduced from purely classical considerations. The detailed computation for the special case of $d = 1$ will be published elsewhere [9]. The result is,

$$f_t(x) = \frac{1}{\sqrt{t/t_0}\Omega_0}\,\hat{f}\left(\frac{\Omega}{\sqrt{t/t_0}\Omega_0}\right) \qquad \text{for } d = 1 \tag{15}$$

where $\Omega_0 = l_0^{d_0}$ is the unit-cell volume. The scaling function $\hat{f}(x)$ satisfies $\int_0^\infty \hat{f}(x)\,{\rm d}x = 1$ and $\int_0^\infty x\,\hat{f}(x)\,{\rm d}x = \sqrt{\pi/2}$. It is plotted in figure 1.

The average volume which is explored by POs of length t will be denoted by $\Omega_e(t)$. For $d = 1$ obviously $\Omega_e(t) \propto \sqrt{t}$, while for $d > 2$ the average volume which is explored after time t is $\Omega_e(t) \sim t$ to leading order. Specifically, one may write $\Omega_e(t) = \ell_0^{d_0}(t/t_0)F(t/t_0)$ where, following [13],

$$F(\tau) = \begin{cases} \sqrt{\dfrac{8}{\pi}}\,\dfrac{1}{\tau^{1/2}} & \text{for } d = 1 \\ \pi\,\dfrac{1}{\ln(\tau)} & \text{for } d = 2 \\ c + \dfrac{c'}{\tau^{(d-2)/2}} & \text{for } 2 < d < 4 \\ c + \dfrac{c''}{\tau} & \text{for } 4 < d. \end{cases} \tag{16}$$

Above c and c' and c'' are constants of order unity (for simple cubic-like structure $c \sim 0.7$). Note that the numerical prefactor $\sqrt{8/\pi}$ for the $d = 1$ case in (16) is somewhat larger than the $\sqrt{\pi/2}$ which is implied by (15). This difference is probably due to the fact that (16) is not an exact result if POs are concerned, rather it is an exact result for wandering trajectories. The transient time t_0 is actually a statistical entity, hence, associated with $\Omega_e(t)$ one should consider a dispersion $\Delta\Omega_e(t) \approx \ell_0^{d_0} \cdot \sqrt{t/t_0}$. Note that in the case of equation (15), the average explored volume and its dispersion are derived from a one-parameter scaling relation. This is not the case for $1 < d$ diffusive system.

It is essential to distinguish the average explored volume $\Omega_e(t)$ from the diffusion volume $\Omega_d(t)$. The latter is determined by the diffusion law $\Omega_d(t) = \ell_0^{d_0-d}(\mathcal{D}_0 t)^{d/2}$. The diffusion volume Ω_d refers to the instantaneous profile of an evolving distribution. It roughly equals the total volume of those cells which are occupied by the evolving distribution. Note that $P_{\rm cl}(t) \approx \Omega/\Omega_d(t)$.

Given the distribution $f_t(\Omega)$, expression (13) can be cast back into the concise form $K(k,t) = C(k,t)K_D(t)$ with

$$C(k,t) = \int_0^\infty f_t(\Omega)\,\mathrm{d}\Omega\,\hat{C}(k\lambda(\Omega,t)). \tag{17}$$

The diffusive behaviour that corresponds to the diagonal approximation prevails as long as the condition $2\pi < k\lambda(\Omega_e(t))$ is satisfied. This condition can be cast into the more suggestive form $t < t_H(\Omega_e(t))$. The equivalence of the latter inequality with the former should be obvious from the discussion of the classical correlation scale in section 4. (There we had taken the reverse route in order to deduce the expression for the classical correlation scale that corresponds to a simple ballistic billiard.) The concept of *running Heisenberg time* $t_H(\Omega_e(t))$ emerges in a natural way from our semiclassical considerations. Originally, this concept was introduced on the basis of a heuristic guess by Allen. In his paper [2] the concept appeared in connection with the tight binding Anderson model where the on-site energies are distributed within range V and the hoping probability is W. There $g_0 \sim W/V$. Allen pointed out that a qualitative agreement with the predictions of scaling theory is recovered if ξ^d in (2) is replaced by $\Omega_e(t)$ as in our formulation. The condition $t < t_H(\Omega_e(t))$ for having a diffusive-like behaviour can be cast into the form $F(t/t_0) > 1/g_0$. It is easily verified that for both $d = 1$ and $d = 2$ the results for the breaktime are consistent with (4) and (5). For $d = 3$ the existence of critical conductance $g_c = c^{-1}$ is a natural consequence, but the exponent in (6) is $2/(d-2)$ rather than νd.

Figure 2 illustrates the different time regimes for quantal evolution versus disorder for $d = 1, 2, 3$. These diagrams constitute an improvement† over those of [14] and [6]. For 'zero disorder' t_0 may be interpreted as the ballistic timescale that corresponds to the shortest PO. The breaktime is volume limited and determined by the Heisenberg time (1). As the disorder grows larger, two distinct classical timescales emerge, now t_0 is the ergodic time with respect to one cell, and $t_{\rm erg}$ is the actual time for ergodicity over the whole volume. The latter is determined by the diffusion coefficient as in (14). If the disorder is not too large, the breaktime is still limited by the Heisenberg time. Going to the other extreme limit of very large disorder ($g_0 < 1$) is not very interesting since the particle will be localized within the volume of a single cell. For weaker disorder there is a crossover from a diffusive-like behaviour (which is actually anomalous for $d = 3$) to localization. The crossover time is determined by equations (4)–(6). In one dimension the crossover

† Compared with [6] the main differences are: The $d = 2$ diagram should be distinguished from the $d = 1$ diagram since the former should include an intermediate regime; the role of Heisenberg time for $d = 3$ should be manifest in the ohmic regime; the localization axis should be appropriately placed.

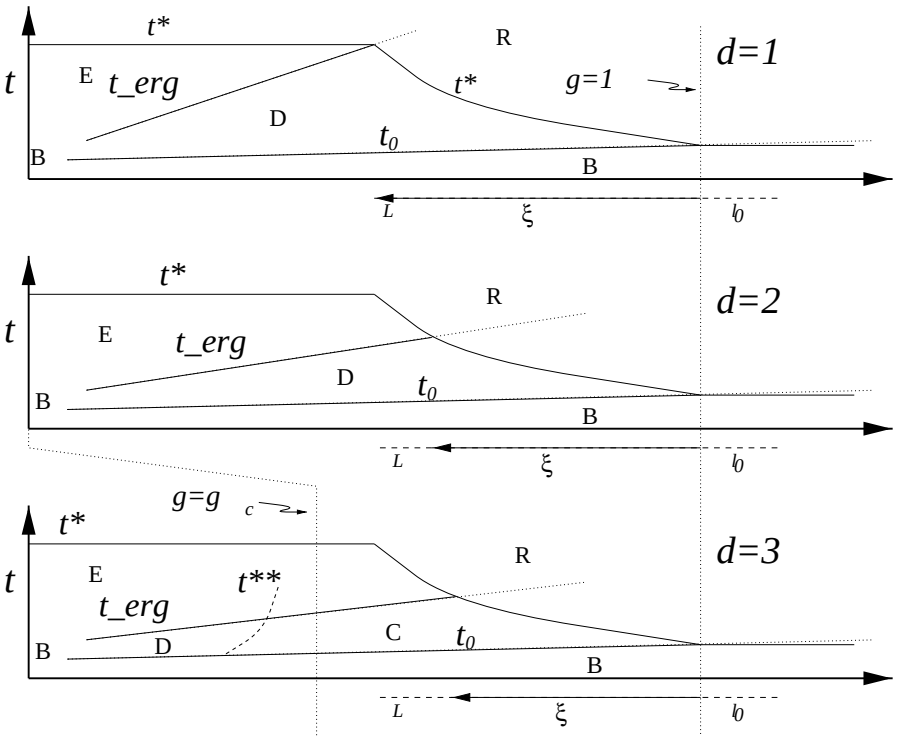

Figure 2. The different time regimes for quantal evolution (vertical axis) versus disorder (horizontal axis), for one-two-three dimensional system. The ballistic (B), diffusive (D), critical (C), ergodic (E) and Recurrence (R) regimes are labelled. See further explanations of these diagrams in the text.

from 'Heisenberg-limited' breaktime to 'disorder-limited' breaktime happens to coincide with the classical curve for $t_{\rm erg}$. This coincidence does not occur in $d = 2$ and therefore we have an intermediate regime where the breaktime occurs *after* ergodization, but is still disorder limited rather than volume limited. In three dimensions we have a qualitatively new regime $g_c < g_0$ where a purely diffusive (ohmic) behaviour (rather than diffusive-like behaviour) prevails. The breaktime here is volume limited. Still, the border between the ohmic regime and the so-called 'critical' one is non-trivial. Scaling theory predicts that the ohmic behaviour is set only after a transient time t^{**} which is given by (6) with $g_c < g_0$. In order to give a semiclassical explanation for t^{**} we should refine somewhat our argumentation. The condition for *purely ohmic* behaviour becomes $t < t_H(\Omega_e(t) - \Delta\Omega_e(t))$. If g_0 is close to g_c then there will appear a transient time $t < t^{**}$ where the *bare* diagonal approximation is unsatisfactory. Note, however, that the critical exponent turns out to be, by this argumentation, 2 rather than νd.

6. The BLC approximation scheme

We focus our attention on the actual computation of the form factor $K(\tau)$. Irrespective of any particular assumption it is easily verified that $K(t_0) = t_H/g_0$, while the asymptotic value $K_{\rm qm}(t) = t_H$ should be obtained for sufficiently long time. The asymptotic behaviour reflects the discrete nature of the quantal spectrum. This feature imposes a major restriction

on the functional form of $\hat{C}(\kappa)$. Using $\hat{C}_{\rm ballistic}(\kappa)$ (see the discussion after (12)) one obtains that for a *simple* ballistic billiard there is a scaling function such that $K(t) = \Omega \hat{K}_0(t/\Omega)$, where Ω is the total volume. The correct asymptotic behaviour is trivially obtained since by construction $\hat{C}_{\rm ballistic}(\kappa)$ gives the correct quantum mechanical result.

For a disordered quasi-one-dimensional billiard, in order to determine the form factor, we should substitute (15) into (17). However, also the scaling function $\hat{C}(\kappa)$ should be specified. In order to make further progress towards a quantitative theory let us assume that it is simply equal to $\hat{C}_{\rm ballistic}(\kappa)$. Using this assumption of 'ballistic-like correlations' (BLC) one obtains that the form factor satisfies the expected scaling property $K(t) = \Omega \hat{K}_d(t)$. The latter scaling property, which implies the existence of a characteristic scaling function $\hat{K}_d(\tau)$, distinguishes a system with localization. Using the BLC approximation scheme the calculated $\hat{K}_1(\tau)$ is

$$\hat{K}_1(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty {\rm d}x \; f\left(\frac{x}{\sqrt{\tau}}\right) \frac{x}{\tau} \hat{K}_0\left(\frac{\tau}{x}\right). \tag{18}$$

This is plotted in figure 1. Indeed the breaktime is disorder limited rather than volume limited. However, one observes that the computation yields the asymptotic behaviour $K_{\rm qm}(t) = t_H/2$ rather than $K_{\rm qm}(t) = t_H$. This implies that the classical correlations have been overestimated by the BLC approximation scheme.

The BLC approximation scheme can be applied for the analysis of $1 < d$ localization. As in the $d = 1$ case, the correct asymptotic behaviour is *not* obtained. The only way to guarantee a correct asymptotic behaviour is to conjecture that $\hat{C}(\kappa) = (\Omega_d(t)/\Omega_e(t))\hat{C}_{\rm ballistic}(\kappa)$ for $\kappa < 1$. This required assumption implies that despite the net repulsion, the classical spectrum is further characterized by strong clustering. The effective clustering may be interpreted as arising from leaking of POs via 'transverse' holes, thus leaving out bundles of POs. Note that the normalization of $p(x)$ does not hold due to the leaking. Therefore $\hat{C}(\kappa)$ is modified in a way that is not completely compatible with its ballistic scaling form.

For completeness we note that in the 'critical regime' of $d = 3$ localization, it has been suggested [14] to put by hand the information concerning the anomalous sub-diffusive behaviour known from scaling theory. One obtains '$P_{\rm cl}(t)$' $\sim (\mathcal{L}/\ell_0)^d(t_0/t)$ and hence $K(t)$ is essentially the same as for $d = 2$ system. The semiclassical justification for this procedure is not clear (see, however, [15]). We believe that a better strategy would be to find the functional form of $f_t(\Omega)$ and $\hat{C}(\kappa)$ and to use (17).

7. Concluding remarks

We have demonstrated that simple semiclassical considerations are capable of giving an explanation for the existence of a disordered-limited breaktime. Qualitatively, the results for the breaktime were in agreement with those of the scaling theory of localization. In section 6 we have briefly discussed the question whether future quantitative semiclassical theory for localization is feasible. It turns out that the simplest (BLC) approximation scheme overestimates the rigidity of the classical spectrum.

Acknowledgments

I would like to thank Harel Primack and Uzy Smilansky for interesting discussions. Thomas Dittrich and Shmuel Fishman are acknowledged for their comments. One of the referees is

acknowledged for contributing the first paragraph of this paper. This research was supported by the Minerva Center for Nonlinear Physics of Complex systems.

References

[1] Argaman N, Dittes F M, Doron E, Keating J P, Kitaev A Y, Sieber M and Smilansky U 1993 *Phys. Rev. Lett.* **71** 4326
J P Keating 1993 *Proc. Int. School of Physics 'Enrico Fermi', Course CXIX* ed G Casati *et al* (Amsterdam: North-Holland)
[2] Allen P B 1980 *J. Phys. C: Solid State Phys.* **13** L667
[3] Sepulveda M A, Tomsovic S and Heller E J 1992 *Phys. Rev. Lett.* **69** 402
[4] Primack H *et al* 1996 *Phys. Rev. Lett.* **76** 1615
[5] Berry M V 1985 *Proc. R. Soc.* A **400** 229
[6] Dittrich T 1996 *Phys. Rep.* **271** 267
[7] Chirikov B V, Izrailev F M and Shepelyansky D 1981 *Sov. Sci. Rev.* C **2** 209
[8] Scharf R and Sundaram B 1996 *Phys. Rev. Lett.* **76** 4907
[9] Cohen D, Primack H and Smilansky U 1997 Quantal-classical duality and the SCTF *Ann. Phys.* Preprint chao-dyn/9708017
[10] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
For general review see Imry Y 1997 *Introduction to Mesoscopic Physics* (Oxford: Oxford University Press)
[11] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[12] Doron E, Smilansky U and Dittrich T 1992 *Physica* **179B** 1
Argaman N, Smilansky U and Imry Y 1993 *Phys. Rev.* B **47** 4440
[13] Montroll E W and Weiss G H 1965 *J. Math. Phys.* **6** 167
[14] Aronov A G, Kravtsov V E and Lerner I V 1995 *Phys. Rev. Lett.* **74** 1174
[15] Chalker J T, Lerner I V and Smith R A 1996 *Phys. Rev. Lett.* **77** 554